Tetrahedra, octahedra and cubo-octahedra: integrable geometry of multi-ratios

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Abstract

Geometric and algebraic aspects of multi-ratios M_{2n} are investigated in detail. Connections with Menelaus' theorem, Clifford configurations and Maxwell's reciprocal quadrangles are utilized to associate the multi-ratios $\mathsf{M}_4, \mathsf{M}_6$ and M_8 with tetrahedra, octahedra and cubo-octahedra respectively. Integrable maps defined on face-centred (fcc) lattices and irregular lattices composed of the face centres of simple cubic lattices are constructed and related to the discrete KP and BKP equations and the integrable discrete Darboux system governing conjugate lattices. An interpretation in terms of integrable irregular lattices of slopes on the plane is also given.

1 Introduction

The classical cross-ratio in conformal and projective geometry [9, 36] has been the subject of extensive studies. Apart from its significance in pure (differential) geometry, it also finds application in other areas such as the geometric analysis of flows in fluid mechanics [19, 38]. In the past decade, the importance of the cross-ratio and its natural quaternionic generalization has been recognized in the field of discrete integrable geometry in the context of discrete holomorphic functions and conformal mappings, circle and sphere patterns, and discrete isothermic, constant mean curvature and minimal surfaces (see [3] and references therein).

The multi-ratio of $2n$ points on the complex plane (M_{2n}) which constitutes a canonical algebraic extension of the cross-ratio appears to have attracted much less attention even though a geometric interpretation of real and purely imaginary multi-ratios was given in as early as 1937 by Morley and Musselman [32]. Multi-ratios may be used in the formulation of Carnot's theorem [8] and canonical generalizations of Ceva's and Menelaus' theorems [28, 39]. They appear naturally in the context of the integrable Gaudin spin model [17, 18]. 'Elliptic' versions of multi-ratios make an appearance in integrable time discretizations of the Calogero-Moser and Ruijsenaars-Schneider models [33, 34] and in statistical mechanics in connection with the Bethe ansatz [26, 27]. Only recently, the six-point multi-ratio condition $\mathsf{M}_6 = -1$ has been employed in the study of hexagonal circle patterns, regular triangular lattices and symmetric circle patterns [1, 2].

In [24], a remarkable connection between Menelaus' theorem of plane geometry, the multi-ratio condition $\mathsf{M}_6 = -1$ and the integrable Kadomtsev-Petviashvili (KP) hierarchy of soliton equations has been brought to light. In a subsequent paper [25], Maxwell's reciprocal figures of graphical statics representing frameworks in equilibrium have been linked to the KP hierarchy of B-type and cross-ratio relations involving M_4. In both cases, it has been demonstrated that the underlying discrete KP and BKP equations constitute canonical objects of plane inversive geometry.

Here, we further explore the geometry and algebra of multi-ratios. We present a novel characterization of reciprocal quadrangles and establish a formal connection with Menelaus' theorem via the multi-ratio condition $\mathsf{M}_6 = -1$. We show that BKP lattices consisting of an infinite number of reciprocal quadrangles are governed by the multi-ratio condition $\mathsf{M}_6 = -1$ and three multi-ratio conditions $\mathsf{M}_8 = 1$. Interestingly, the latter three conditions may also be interpreted as an integrable Möbius invariant version of the discrete Darboux system descriptive of conjugate lattices. We demonstrate that the multi-ratio condition $\mathsf{M}_6 = -1$ constitutes an admissible constraint which 'propagates' through the lattice.

In [25], a natural correspondence between the cross-ratio M_4 and tetrahedra has been recorded. In the present paper, this connection is reiterated and the multi-ratios M_6 and M_8 are shown to be canonically defined on octahedra and cubo-octahedra respectively. The vertices of these (quasi-)regular polyhedra are interpreted as the vertices of regular face-centred cubic (fcc) lattices and irregular lattices composed of the face centres of simple cubic lattices. A well-posed Cauchy problem for BKP lattices and associated irregular lattices of slopes on the plane is formulated.

2 Menelaus' theorem and reciprocal quadrangles

The present paper is based on an interesting observation which characterizes quadrangles and provides a link with Menelaus' classical theorem of plane geometry. Here, we briefly review the basic geometric and algebraic properties of Menelaus figures and reciprocal quadrangles [24, 25] and record the abovementioned novel characterization of quadrangles.

2.1 Menelaus' theorem

We begin with a fundamental theorem of ancient Greek geometry which bears the name of Menelaus but may have been known to Euclid [9, 36].

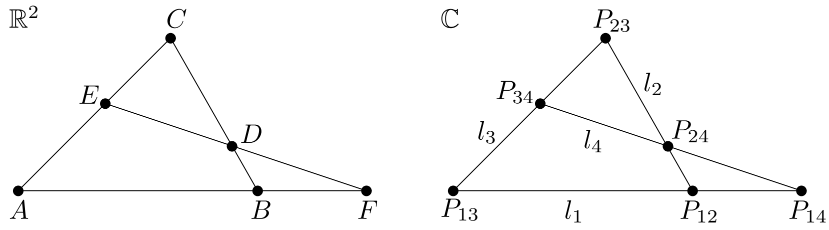

Figure 1: A Menelaus figure

Theorem 1 (Menelaus' theorem and its converse). *Let A, B, C be the vertices of a triangle and D, E, F be three points on the (extended) edges of the triangle opposite A, B, C respectively (see Figure 1). Then, the points D, E, F are collinear if and only if*

$$\frac{\overline{AF}}{\overline{FB}}\frac{\overline{BD}}{\overline{DC}}\frac{\overline{CE}}{\overline{EA}} = -1, \tag{2.1}$$

where $\overline{PQ}/\overline{QR}$ denotes the ratio of directed lengths associated with any three collinear points P, Q, R.

For our purposes, it is convenient to regard the plane as the complex plane and label the points by complex numbers as indicated in Figure 1. Thus, if we define the multi-ratio of $2n$ complex numbers by

$$\mathsf{M}_{2n} = \mathsf{M}(P_1, \ldots, P_{2n}) = \frac{(P_1 - P_2)(P_3 - P_4)\cdots(P_{2n-1} - P_{2n})}{(P_2 - P_3)(P_4 - P_5)\cdots(P_{2n} - P_1)} \tag{2.2}$$

then the Menelaus relation (2.1) assumes the form

$$\mathsf{M}(P_{13}, P_{14}, P_{12}, P_{24}, P_{23}, P_{34}) = -1. \tag{2.3}$$

It is important to note that the multi-ratio is invariant under the group of Möbius transformations acting on the complex plane. This property is well-known for the classical cross-ratio of four points [9, 36] which is represented by $n = 2$.

It is evident that the above multi-ratio relation may also be regarded as an identity for the six points of intersection of four generic straight lines on the complex plane. However, in general, the geometry of the multi-ratio condition (2.3) is that of four circles S_1, S_2, S_3, S_4 meeting at a point P as displayed in Figure 2. Indeed, if we label the point of intersection of two circles S_i and S_k by P_{ik} then the following theorem obtains [24]:

Theorem 2 (The geometry of the multi-ratio condition $\mathsf{M}_6 = -1$). *Four generic circles S_1, S_2, S_3, S_4 on the complex plane pass through a point P if and only if the points of intersection $P_{13}, P_{14}, P_{12}, P_{24}, P_{23}, P_{34}$ satisfy the multi-ratio condition*

$$\mathsf{M}(P_{13}, P_{14}, P_{12}, P_{24}, P_{23}, P_{34}) = -1. \tag{2.4}$$

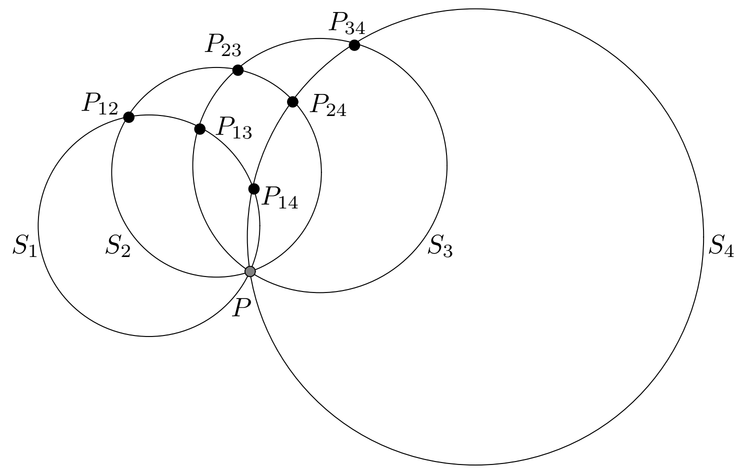

Figure 2: A Menelaus configuration

The connection with Menelaus' theorem is provided by the symmetry group of the multi-ratio condition (2.4). Indeed, since the latter is invariant under the group of inversive transformations [9, 36], that is Möbius transformations and complex conjugation, the point P may be mapped to infinity by means of an inversive transformation without changing the multi-ratio. The circles S_1, S_2, S_3, S_4 then become straight lines l_1, l_2, l_3, l_4 and a Menelaus figure is obtained (cf. Figure 1).

Theorem 2 implies that the multi-ratio condition (2.4) is invariant under any permutation of the indices $1, 2, 3, 4$. However, the multi-ratio condition (2.4) may also be formulated as

$$\mathsf{M}(P_{14}, P_{12}, P_{24}, P_{23}, P_{34}, P_{13}) = -1. \tag{2.5}$$

This implies that the four circles S_{ikl} passing through the points P_{ik}, P_{il}, P_{kl} also meet at a point P_{1234}, say (see Figure 3). This is the content of a classical theorem due to Clifford [10] and, indeed, point-circle configurations of this kind are known as Clifford configurations [40].

2.2 Reciprocal quadrangles

In the preceding, we have discussed the properties of figures consisting of four lines and six points and their inversive geometric generalization. Here, we focus on figures involving six lines and four points, namely quadrangles. Thus, given four points $\Phi_{23}, \Phi_{13}, \Phi_{12}, \Phi$ on the complex plane, we denote the six lines linking these points by $\alpha, \beta, \gamma, \alpha_1, \beta_2, \gamma_3$ as indicated in Figure 4. We may now inquire as to whether there exists another quadrangle which is such that the lines of the

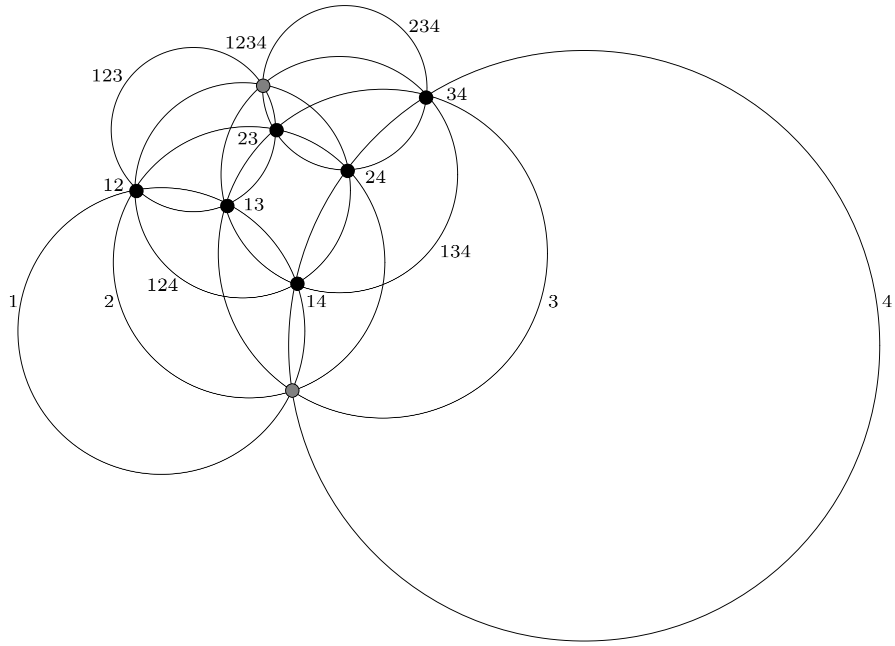

Figure 3: A $\mathcal{C}_4$ Clifford configuration

two quadrangles are pairwise parallel and any three lines meeting at a point of one quadrangle correspond to the edges of a triangle in the other (see Figure 4). It turns out that such a quadrangle always exists and is uniquely defined up to a scaling. Since the defining relation between the two quadrangles is reciprocal in nature, the two quadrangles $(\Phi_{23}, \Phi_{13}, \Phi_{12}, \Phi)$ and $(\Phi_1, \Phi_2, \Phi_3, \Phi_{123})$ are termed reciprocal quadrangles.†

In 1864, Maxwell [30] gave a remarkably simple constructive proof of the existence of reciprocal quadrangles. His investigation of reciprocal quadrangles and, more generally, reciprocal figures (configurations of points and lines) was instigated by a problem of graphical statics, namely the existence of frames which can support forces. Thus, if a reciprocal figure exists then the original figure (e.g. a quadrangle) may be regarded as a frame which is in equilibrium with closed diagrams of forces provided by the closed polygons (e.g. triangles) in the reciprocal figure. Reciprocal figures and related graphical methods were applied extensively by Maxwell's contemporaries in [7, 11, 12, 20, 21, 37]. Moreover, Maxwell discovered a fascinating connection between the existence of reciprocal figures and the representation of reciprocal figures as orthogonal projections of closed polyhedra. Interestingly, a century later, this connection was rediscovered

†In [25], we referred to these as *reciprocal triangles* since the points Φ and Φ_{123} are uniquely determined by the triangles $(\Phi_{23}, \Phi_{13}, \Phi_{12})$ and (Φ_1, Φ_2, Φ_3). Here, we prefer to use the classical term *quadrangle*.

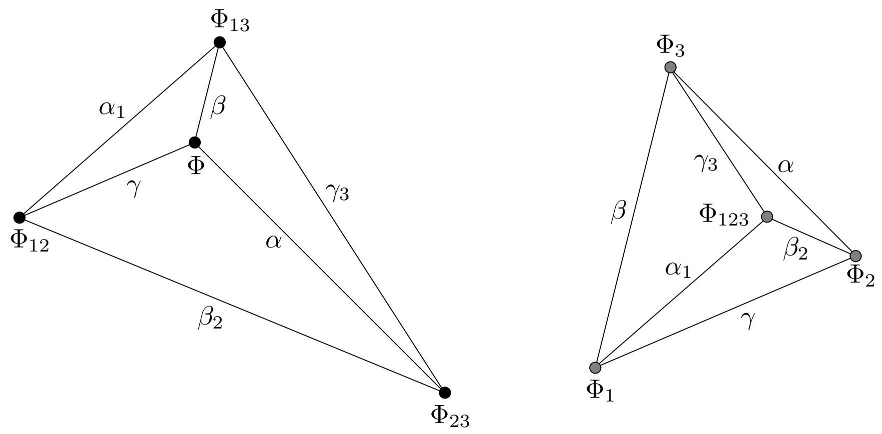

Figure 4: Reciprocal quadrangles

in the context of 'artificial intelligence', namely the recognition and realizability of plane line drawings as three-dimensional polyhedral scences [16, 29].

It turns out [25] that the vertices of reciprocal quadrangles obey the cross-ratio relation

$$\mathsf{M}(\Phi_{23}, \Phi_{13}, \Phi_{12}, \Phi) = \mathsf{M}(\Phi_1, \Phi_2, \Phi_3, \Phi_{123}). \tag{2.6}$$

This relation is preserved by Möbius transformations which act independently on the two quadrangles. Either quadrangle is thereby mapped to a quadrangle whose vertices are linked by circular arcs which meet at a point. In [25], this observation has been exploited to define reciprocal quadrangles in the setting of inversive geometry and it has been shown that these are governed by the cross-ratio relation (2.6).

The novel key observation is now the following: if we regard the labels $\alpha, \beta, \gamma, \alpha_1, \beta_2, \gamma_3$ as the slopes of the edges of the quadrangle $(\Phi_{23}, \Phi_{13}, \Phi_{12}, \Phi)$ then it is not difficult to show that

$$\mathsf{M}(\alpha, \beta_2, \gamma, \alpha_1, \beta, \gamma_3) = -1. \tag{2.7}$$

A proof of this fact will be obtained as a by-product of the deliberations of Section 4. In fact, it will be shown that (2.7) is also a *sufficient* condition for six lines being part of a quadrangle. The multi-ratio condition (2.7) written as

$$\mathsf{M}(\alpha_1, \beta, \gamma_3, \alpha, \beta_2, \gamma) = -1 \tag{2.8}$$

therefore implies that there exists another quadrangle $(\Phi_1, \Phi_2, \Phi_3, \Phi_{123})$ with the same slopes but different incidence structure (cf. Figure 4). Thus, an elegant alternative proof of the existence of reciprocal quadrangles has been established. This is summarized in the following theorem:

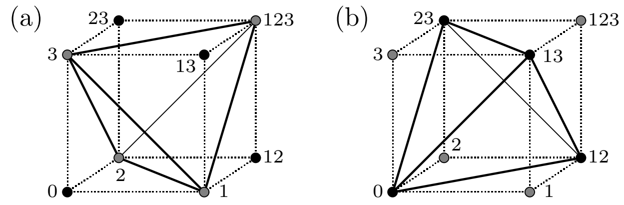

Figure 5: The elementary cells of $G^{(1)}$ and $G^{(0)}$

Theorem 3 (A characterization of reciprocal quadrangles). *Six lines with slopes $\alpha, \beta, \gamma, \alpha_1, \beta_2, \gamma_3$ are parallel to the edges of reciprocal quadrangles of the incidence structure displayed in Figure 4 if and only if*

$$\mathsf{M}(\alpha, \beta_2, \gamma, \alpha_1, \beta, \gamma_3) = -1. \tag{2.9}$$

3 Integrable lattices associated with Menelaus configurations and reciprocal quadrangles

In [24, 25], it has been shown that there exist canonical integrable lattices on the complex plane which encapsulate an infinite number of Menelaus figures, Clifford configurations and reciprocal quadrangles. Their construction is summarized in this section.

3.1 Menelaus and Clifford lattices

We first observe that it is canonical to interpret the lines l_1, l_2, l_3, l_4 in the Menelaus figure 1 as degenerate triangles. In this way, we may think of a Menelaus figure as consisting of eight triangles and six vertices. Accordingly, a Menelaus figure admits the same combinatorics as an octahedron. Indeed, this becomes evident if one inspects the Clifford configuration displayed in Figure 3. Thus, the six points P_{ik} represent the vertices of an octahedron while the eight circles are associated with the eight triangular faces of the octahedron. We may therefore regard a Menelaus figure or a Clifford configuration as the image of an octahedron under some map $\boldsymbol{\psi}$, say, which preserves the combinatorics.

It is natural to consider maps from a face-centred cubic (fcc) lattice to the complex plane, that is

$$\begin{aligned} &\boldsymbol{\psi} : G^{(1)} \to \mathbb{C} \\ &\qquad G^{(1)} = \{(n_1, n_2, n_3) \in \mathbb{Z}^3 : n_1 + n_2 + n_3 \text{ odd}\}. \end{aligned} \tag{3.1}$$

The edge structure of $G^{(1)}$ is obtained by starting at the vertex $(1, 0, 0)$ of the $\mathbb{Z}^3$ simple cubic lattice and drawing diagonals across the faces of the cubes. The six diagonals on each cube then form a tetrahedron as shown in Figure 5(a). The

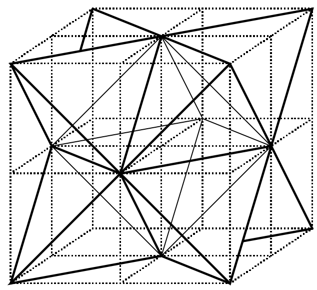

Figure 6: The stella octangula

eight tetrahedra inscribed in any eight adjacent cubes enclose an octahedron (cf. Figure 6), thereby forming a stella octangula, while any tetrahedron is enclosed by four octahedra. Thus, the $G^{(1)}$ fcc lattice is composed of tetrahedra and octahedra. We now demand that any octahedron be mapped to a Clifford configuration in the sense specified earlier. Accordingly, the map $\boldsymbol{\psi}$ is defined by the multi-ratio condition

$$\mathsf{M}(\boldsymbol{\psi}_{\bar{1}}, \boldsymbol{\psi}_2, \boldsymbol{\psi}_{\bar{3}}, \boldsymbol{\psi}_1, \boldsymbol{\psi}_{\bar{2}}, \boldsymbol{\psi}_3) = -1, \tag{3.2}$$

where the arguments of $\boldsymbol{\psi}$ have been suppressed and the notation

$$\boldsymbol{\psi} = \boldsymbol{\psi}(n_1, n_2, n_3), \quad \boldsymbol{\psi}_{\bar{1}} = \boldsymbol{\psi}(n_1-1, n_2, n_3), \quad \boldsymbol{\psi}_1 = \boldsymbol{\psi}(n_1+1, n_2, n_3), \ldots \tag{3.3}$$

has been used.

The multi-ratio condition (3.2) which now constitutes a lattice equation is nothing but an integrable discrete version of the Schwarzian Kadomtsev-Petviashvili (SKP) equation and, in fact, its entire hierarchy [5, 6]. It may also be regarded as a superposition principle for solutions of the Schwarzian KP hierarchy as well as a permutability theorem associated with Darboux-type transformations [24]. Since the dSKP equation is equivalent to the complex discrete KP equation

$$\tau_{\bar{1}}\tau_1 + \tau_{\bar{2}}\tau_2 + \tau_{\bar{3}}\tau_3 = 0, \tag{3.4}$$

there exists a remarkable connection between Menelaus' theorem and Hirota's 'master equation' (3.4). Links with pseudo-analytic functions (quasi-conformal mappings) have also been recorded in [24]. We observe in passing that real solutions of the 'tau-function' equation (3.4) correspond to Menelaus lattices on the complex plane, that is lattices which embody Menelaus figures rather than Clifford configurations.

3.2 Lattices composed of reciprocal quadrangles

It is evident that quadrangles admit the combinatorics of tetrahedra. Indeed, as Maxwell pointed out [30], the very fact that a quadrangle may be regarded as an orthogonal projection of a tetrahedron guarantees the existence of a reciprocal quadrangle. Thus, we now think of a quadrangle as the image of a tetrahedron under a map Φ, say, which preserves the combinatorics. A lattice composed of quadrangles is obtained by mapping the tetrahedra of an fcc lattice onto the complex plane. If we consider two such maps, we may demand that the images of corresponding pairs of tetrahedra constitute reciprocal quadrangles.

Under the assumption of a natural correspondence between pairs of tetrahedra, the integrability of maps of the afore-mentioned kind has been established in [25]. Indeed, if, for convenience, we choose the vertices of the second fcc lattice $G^{(0)}$ as the complement of $G^{(1)}$ with respect to $\mathbb{Z}^3$, that is (cf. Figure 5(b))

$$G^{(0)} = \{(n_1, n_2, n_3) \in \mathbb{Z}^3 : n_1 + n_2 + n_3 \text{ even}\}, \tag{3.5}$$

then we may combine the two maps in question to

$$\Phi : \mathbb{Z}^3 \to \mathbb{C} \tag{3.6}$$

and demand that the images of the two tetrahedra in any elementary cube of the $\mathbb{Z}^3$ lattice under Φ be reciprocal quadrangles. If, as usual, indices indicate increments of the respective variables then the pairs of reciprocal quadrangles are labelled as in Figure 4, namely by $(\Phi_{23}, \Phi_{13}, \Phi_{12}, \Phi)$, $(\Phi_1, \Phi_2, \Phi_3, \Phi_{123})$.

In order to provide an analytic description of lattices on the complex plane which consist of reciprocal quadrangles, it is convenient to focus initially on one pair of reciprocal quadrangles. Thus, eight points $\Phi, \ldots, \Phi_{123}$ on the complex plane constitute the vertices of two reciprocal quadrangles $(\Phi_{23}, \Phi_{13}, \Phi_{12}, \Phi)$, $(\Phi_1, \Phi_2, \Phi_3, \Phi_{123})$ if and only if there exist six real dilation coefficients a, b, c and a_1, b_2, c_3 such that

$$\begin{aligned} \Phi_{12} - \Phi &= c(\Phi_1 - \Phi_2) \\ \Phi_{23} - \Phi &= a(\Phi_2 - \Phi_3) \\ \Phi_{13} - \Phi &= b(\Phi_3 - \Phi_1) \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} \Phi_{123} - \Phi_3 &= c_3(\Phi_{13} - \Phi_{23}) \\ \Phi_{123} - \Phi_1 &= a_1(\Phi_{12} - \Phi_{13}) \\ \Phi_{123} - \Phi_2 &= b_2(\Phi_{23} - \Phi_{12}). \end{aligned} \tag{3.8}$$

As shown in [25], on use of (3.7), elimination of Φ_{123} from (3.8) leads to three relations for the dilation coefficients only, viz

$$a_1 = -\frac{a}{ab + bc + ca}, \quad b_2 = -\frac{b}{ab + bc + ca}, \quad c_3 = -\frac{c}{ab + bc + ca}. \tag{3.9}$$

This result may be interpreted in two ways. Firstly, it provides an algebraic proof of the existence of reciprocal quadrangles in that if we choose an arbitrary quadrangle $(\Phi_{23}, \Phi_{13}, \Phi_{12}, \Phi)$ and arbitrarily prescribe three real dilation coefficients a, b, c then the system (3.7), (3.8) determines a reciprocal quadrangle $(\Phi_1, \Phi_2, \Phi_3, \Phi_{123})$ uniquely up to translation. Secondly, in the case of maps Φ which are to encode an infinite number of reciprocal quadrangles, the dilation coefficients a, b, c are functions of the discrete variables and the three linear equations (3.8) are nothing but incremented versions of the three linear equations (3.7). The nonlinear system (3.9) then represents the compatibility conditions which guarantee the existence of the map Φ. Thus, any solution of the discrete system (3.9) gives rise to a map Φ which may be decomposed into two maps defined on the complementary fcc lattices with the required properties.

The relations $a_1 b = b_2 a$, $b_2 c = c_3 b$ and $c_3 a = a_1 c$ imply the existence of a potential τ which parametrizes the dilations a, b, c according to

$$a = \frac{\tau_2\tau_3}{\tau\tau_{23}}, \quad b = \frac{\tau_1\tau_3}{\tau\tau_{13}}, \quad c = \frac{\tau_1\tau_2}{\tau\tau_{12}} \tag{3.10}$$

so that (3.9) reduces to the integrable discrete BKP (dBKP) equation

$$\tau\tau_{123} + \tau_1\tau_{23} + \tau_2\tau_{13} + \tau_3\tau_{12} = 0. \tag{3.11}$$

We therefore refer to the lattices on the complex plane defined by Φ as BKP lattices. The dBKP equation is known to discretize the complete BKP hierarchy of soliton equations [31]. It also represents a superposition principle for eight solutions of a 2+1-dimensional sine-Gordon system [22] generated by the classical Moutard transformation [35]. Thus, once again, there exists a remarkable connection between reciprocal figures of graphical statics and the important Miwa equation (3.11).

4 A novel characterization of BKP lattices. Integrable irregular lattices of slopes on the plane

In Section 2, it has been stated that reciprocal quadrangles may be characterized by a constraint on the slopes of their edges, namely $\mathsf{M}_6 = -1$. Here, we give a proof of this assertion and derive the necessary and sufficient conditions on the slopes which guarantee the existence of lattices composed of reciprocal quadrangles.

4.1 Slope characterization of reciprocal quadrangles

We first decompose any point Φ on the complex plane in its real and imaginary parts according to $\Phi = \varphi + \mathrm{i}\psi$ and note that six points $\Phi, \ldots, \Phi_{123}$ constitute the vertices of two reciprocal quadrangles $(\Phi_{23}, \Phi_{13}, \Phi_{12}, \Phi)$, $(\Phi_1, \Phi_2, \Phi_3, \Phi_{123})$ if and only if the relations

$$\begin{aligned}
\frac{\varphi_{12}-\varphi}{\varphi_1-\varphi_2} &= \frac{\psi_{12}-\psi}{\psi_1-\psi_2}, & \frac{\varphi_{123}-\varphi_3}{\varphi_{13}-\varphi_{23}} &= \frac{\psi_{123}-\psi_3}{\psi_{13}-\psi_{23}}\\
\frac{\varphi_{23}-\varphi}{\varphi_2-\varphi_3} &= \frac{\psi_{23}-\psi}{\psi_2-\psi_3}, & \frac{\varphi_{123}-\varphi_1}{\varphi_{12}-\varphi_{13}} &= \frac{\psi_{123}-\psi_1}{\psi_{12}-\psi_{13}}\\
\frac{\varphi_{13}-\varphi}{\varphi_3-\varphi_1} &= \frac{\psi_{13}-\psi}{\psi_3-\psi_1}, & \frac{\varphi_{123}-\varphi_2}{\varphi_{23}-\varphi_{12}} &= \frac{\psi_{123}-\psi_2}{\psi_{23}-\psi_{12}}
\end{aligned} \tag{4.1}$$

hold. Indeed, if we denote the above ratios by c, c_3, a, a_1, b, b_2 respectively then the linear system (3.7), (3.8) is obtained and the coefficients admit the interpretation as dilations. Alternatively, we may 'linearize' the above relations by introducing coefficients $\alpha, \beta, \gamma, \alpha_1, \beta_2, \gamma_3$ according to

$$\begin{aligned} \psi_{12} - \psi &= \gamma(\varphi_{12} - \varphi), &\quad \psi_{13} - \psi_{23} &= \gamma_3(\varphi_{13} - \varphi_{23}) \\ \psi_{23} - \psi &= \alpha(\varphi_{23} - \varphi), &\quad \psi_{12} - \psi_{13} &= \alpha_1(\varphi_{12} - \varphi_{13}) \\ \psi_{13} - \psi &= \beta(\varphi_{13} - \varphi), &\quad \psi_{23} - \psi_{12} &= \beta_2(\varphi_{23} - \varphi_{12}) \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} \psi_1 - \psi_2 &= \gamma(\varphi_1 - \varphi_2), &\quad \psi_{123} - \psi_3 &= \gamma_3(\varphi_{123} - \varphi_3) \\ \psi_2 - \psi_3 &= \alpha(\varphi_2 - \varphi_3), &\quad \psi_{123} - \psi_1 &= \alpha_1(\varphi_{123} - \varphi_1) \\ \psi_3 - \psi_1 &= \beta(\varphi_3 - \varphi_1), &\quad \psi_{123} - \psi_2 &= \beta_2(\varphi_{123} - \varphi_2). \end{aligned} \tag{4.3}$$

In this case, the coefficients $\alpha, \ldots, \gamma_3$ are but the slopes of the edges of the reciprocal quadrangles displayed in Figure 4. It is emphasized that, by construction, the systems (3.7), (3.8) and (4.2), (4.3) are equivalent.

If we set aside the second system (4.3) then the first system (4.2) is descriptive of a single quadrangle and no reference to reciprocity is made. The relations $(4.2)_{1,3,5}$ may be regarded as definitions of ψ_{12}, ψ_{23} and ψ_{13}. Insertion into the remaining relations then produces the homogeneous linear system

$$\begin{pmatrix} 0 & \gamma_3 - \alpha & \beta - \gamma_3 \\ \gamma - \alpha_1 & 0 & \alpha_1 - \beta \\ \beta_2 - \gamma & \alpha - \beta_2 & 0 \end{pmatrix} \begin{pmatrix} \varphi_{12} - \varphi \\ \varphi_{23} - \varphi \\ \varphi_{13} - \varphi \end{pmatrix} = 0. \tag{4.4}$$

Since the quadrangles are assumed to be non-degenerate, the determinant of the above linear system must vanish. Consequently,

$$\mathsf{M}(\alpha, \beta_2, \gamma, \alpha_1, \beta, \gamma_3) = -1 \tag{4.5}$$

constitutes a necessary condition on the slopes for the existence of a quadrangle. Conversely, if the coefficients $\alpha, \ldots, \gamma_3$ satisfy the multi-ratio condition (4.5) and $\Phi = \varphi + \mathrm{i}\psi$, $\varphi_{23} = \Re(\Phi_{23})$ are arbitrarily prescribed then the points $\Phi_{23}, \Phi_{13}, \Phi_{12}, \Phi$ determined by $(4.2)_{1,3,5}$ and

$$\varphi_{12} - \varphi = \frac{\beta_2 - \alpha}{\beta_2 - \gamma}(\varphi_{23} - \varphi), \quad \varphi_{13} - \varphi = \frac{\gamma_3 - \alpha}{\gamma_3 - \beta}(\varphi_{23} - \varphi) \tag{4.6}$$

constitute the vertices of a reciprocal quadrangle. Moreover, the second linear system (4.3) gives rise to the necessary and sufficient condition

$$\mathsf{M}(\alpha_1, \beta, \gamma_3, \alpha, \beta_2, \gamma) = -1 \tag{4.7}$$

which is equivalent to the multi-ratio condition (4.5). This proves Theorem 3.

It turns out convenient to regard $(4.3)_{1,5}$ as definitions of ψ_1, ψ_2 and $(4.3)_3$ as a constraint on φ_i which may be brought into the two equivalent forms

$$\varphi_1 - \varphi_2 = \frac{\alpha - \beta}{\beta - \gamma}(\varphi_2 - \varphi_3), \quad \varphi_1 - \varphi_3 = \frac{\alpha - \gamma}{\beta - \gamma}(\varphi_2 - \varphi_3). \tag{4.8}$$

Any of the three equations $(4.3)_{2,4,6}$ defines ψ_{123} and the remaining two which are identical modulo the multi-ratio condition (4.7) serve as a definition of φ_{123}. For future reference, we record the two equivalent representations

$$\varphi_{123} - \varphi_2 = \frac{\gamma_3 - \alpha}{\beta_2 - \gamma_3}(\varphi_2 - \varphi_3), \quad \varphi_{123} - \varphi_3 = \frac{\beta_2 - \alpha}{\beta_2 - \gamma_3}(\varphi_2 - \varphi_3). \tag{4.9}$$

4.2 Slope characterization of BKP lattices

Here, as an extension of the preceding, we are concerned with BKP lattices encapsulating an infinite number of reciprocal quadrangles. The fundamental system of lattice equations which is required to hold is now represented by $(4.2)_{1,3,5}$ and $(4.3)_{1,3,5}$. The remaining equations are redundant. The compatibility conditions which guarantee the existence of the function ψ are then given by the lattice system (4.5)-(4.9). It is therefore necessary to establish the consistency of the linear equations (4.6), (4.8) and (4.9). Thus, if we shift (4.8) in the n_3- and n_2-directions respectively then we obtain the expressions

$$\begin{aligned} \varphi_{22} - \varphi_{23} &= \frac{(\beta_2 - \gamma_2)(\gamma - \alpha)}{(\beta_2 - \gamma)(\alpha_2 - \gamma_2)}(\varphi_{23} - \varphi) \\ \varphi_{33} - \varphi_{23} &= \frac{(\gamma_3 - \beta_3)(\beta - \alpha)}{(\gamma_3 - \beta)(\alpha_3 - \beta_3)}(\varphi_{23} - \varphi). \end{aligned} \tag{4.10}$$

Moreover, comparison of (4.9) with (4.6) shifted in the n_3- and n_2-directions respectively leads to

$$\begin{aligned} \varphi_{223} - \varphi_2 &= \frac{(\gamma_{23} - \beta_2)(\gamma_3 - \alpha)}{(\gamma_{23} - \alpha_2)(\beta_2 - \gamma_3)}(\varphi_2 - \varphi_3) \\ \varphi_{233} - \varphi_3 &= \frac{(\beta_{23} - \gamma_3)(\beta_2 - \alpha)}{(\beta_{23} - \alpha_3)(\beta_2 - \gamma_3)}(\varphi_2 - \varphi_3). \end{aligned} \tag{4.11}$$

The relations (4.6) and (4.8)-(4.11) may be cast into the form of a discrete matrix 'Frobenius system', that is it is readily verified that

$$\phi_i = L^{(i)}\phi, \quad \phi = \begin{pmatrix} \varphi \\ \varphi_2 \\ \varphi_3 \\ \varphi_{23} \end{pmatrix}, \tag{4.12}$$

where the matrices $L^{(i)}$, $i = 1, 2, 3$ depend in a known manner on the slopes α, β, γ. Accordingly, the existence of the function φ is guaranteed if and only if the slopes satisfy the compatibility conditions $\phi_{ik} = \phi_{ki}$, that is

$$L_k^{(i)} L^{(k)} = L_i^{(k)} L^{(i)}, \quad i \neq k. \tag{4.13}$$

Apart from the multi-ratio condition (4.5), these imply only three independent constraints on the slopes. A canonical choice is given by the multi-ratio conditions

$$\begin{aligned} \mathsf{M}(\alpha, \beta_2, \alpha_2, \gamma_{23}, \alpha_{23}, \beta_{23}, \alpha_3, \gamma_3) &= 1 \\ \mathsf{M}(\beta, \gamma_3, \beta_3, \alpha_{13}, \beta_{13}, \gamma_{13}, \beta_1, \alpha_1) &= 1 \\ \mathsf{M}(\gamma, \alpha_1, \gamma_1, \beta_{12}, \gamma_{12}, \alpha_{12}, \gamma_2, \beta_2) &= 1. \end{aligned} \tag{4.14}$$

Hence, the following theorem obtains:

Theorem 4 (Slope characterization of BKP lattices). *Three functions α, β, γ may be associated with the slopes of a BKP lattice of reciprocal quadrangles if and only if they obey the multi-ratio conditions (4.5) and (4.14).*

By virtue of (4.2), (4.3) and (4.12), one may immediately observe that a slope function determines a BKP lattice uniquely up to a choice of, for instance, $\varphi, \varphi_2, \varphi_3, \varphi_{23}$ and ψ, ψ_2.

4.3 Geometric implications. Irregular lattices of slopes on the plane

It turns out that the multi-ratio conditions governing BKP lattices may be formulated in a geometrically invariant manner. To this end, let $\Phi : \mathbb{Z}^3 \to \mathbb{C}$ be a BKP lattice. Then, by construction, any two parallel edges of a pair of reciprocal quadrangles constitute the images of the two diagonals of the corresponding face of the cubic lattice $\mathbb{Z}^3$ (cf. Figure 5). It is therefore natural to associate the slope of the two edges with this face. Accordingly, we may regard the slope functions α, β, γ as one function σ, say, which is defined on the centres of the faces of the cubic lattice. The set of centres $\mathbb{F} \subset \mathbb{R}^3$ does not constitute a proper but an 'irregular' lattice since it is not translationally invariant. The image of $\mathbb{F}$ under σ may be thought of as describing an irregular lattice of slopes on the plane. Thus, σ may be identified with a map of the form

$$\sigma : \mathbb{F} \to \mathbb{P}^1, \tag{4.15}$$

where $\mathbb{P}^1$ is a one-dimensional projective space.

We begin with an arbitrary map of the form (4.15). The set of centres $\mathbb{F}$ may be interpreted as the vertices of octahedra inscribed in the elementary cubes of the lattice $\mathbb{Z}^3$ as depicted in Figure 7. We consider a non-planar hexagon $(P_1, P_2, P_3, P_4, P_5, P_6)$ formed by six edges of an octahedron with vertices P_i in such a way that any two adjacent edges of the hexagon belong to a triangular face of the octahedron (cf. Figure 7) and impose the multi-ratio condition

$$\mathsf{M}(\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6) = -1, \tag{4.16}$$

where $\sigma_i = \sigma(P_i)$. Even though there exist four such hexagons, the corresponding multi-ratio conditions $\mathsf{M}_6 = -1$ are equivalent. Thus, we may think of these multi-ratio conditions as one multi-ratio condition

$$\mathsf{M}_{\mathsf{O}} = -1 \tag{4.17}$$

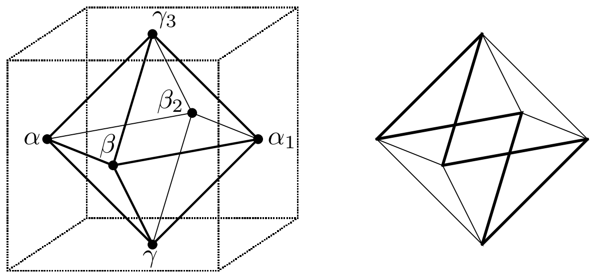

Figure 7: An octahedron and the multi-ratio condition $\mathsf{M}_{\mathsf{O}} = -1$

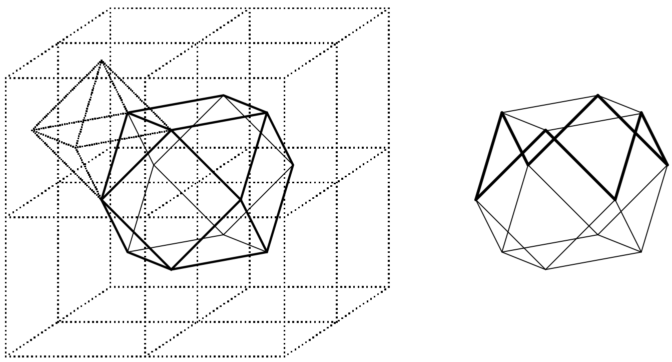

Figure 8: A cubo-octahedron and a multi-ratio condition $\mathsf{M}_{\mathsf{CO}} = 1$

defined on the octahedron. It is evident that the multi-ratio condition (4.5) is equivalent to imposing $\mathsf{M}_{\mathsf{O}} = -1$ on all octahedra.

The eight octahedra inscribed in any eight adjacent elementary cubes of $\mathbb{Z}^3$ are linked to a cubo-octahedron as indicated in Figure 8 so that the set $\mathbb{F}$ is naturally associated with the 'tiling' of Euclidean space by octahedra and cubo-octahedra. Any of the six square faces of a cubo-octahedron is attached to four triangular faces (cf. Figure 8). The edges of the triangles which are not shared by a square form a non-planar octagon $(P_1, P_2, P_3, P_4, P_5, P_6, P_7, P_8)$ on which we may define the multi-ratio condition

$$\mathsf{M}(\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6, \sigma_7, \sigma_8) = 1. \tag{4.18}$$

A straight-forward calculation reveals that any three of the six multi-ratio conditions $\mathsf{M}_8 = 1$ associated with a cubo-octahedron are independent and the remaining three are algebraic consequences. If we denote the multi-ratios defined on a cubo-octahedron by

$$\mathsf{M}_{\mathsf{CO}} = 1 \tag{4.19}$$

then the multi-ratio conditions (4.14) represent $\mathsf{M}_{\mathsf{CO}} = 1$ imposed on all cubo-octahedra. Consequently, Theorem 4 may be reformulated as:

Theorem 5 (Geometrically invariant characterization of BKP lattices). *A map* $\sigma : \mathbb{F} \to \mathbb{P}^1$ *corresponds to a BKP lattice if and only if the multi-ratio conditions* $\mathsf{M}_{\mathsf{O}} = -1$ *and* $\mathsf{M}_{\mathsf{CO}} = 1$ *defined on the octahedra and cubo-octahedra of the irregular lattice* $\mathbb{F}$ *are satisfied.*

It is worth pointing out that the octahedra inscribed in the elementary cubes of a cubic lattice form a subset of the octahedra encapsulated in an fcc lattice. In fact, the results summarized in Section 3 imply that the condition $\mathsf{M}_{\mathsf{O}} = -1$ imposed on the octahedra of an fcc lattice constitutes nothing but the real dSKP equation. Thus, the discrete BKP equation is obtained by considering only the octahedra which belong to the cubic lattice and imposing the additional constraints $\mathsf{M}_{\mathsf{CO}} = 1$. This observation seems reminiscent of the construction of the BKP hierarchy which is obtained from the KP hierarchy by 'freezing' every second flow and imposing symmetry constraints on the remaining flows [14].

We conclude with the remark that, in a similar manner, the cross-ratio M_4 may be related to non-planar quadrilaterals on tetrahedra. Indeed, if we consider a map of the form

$$\Phi : \mathbb{Z}^3 \to \mathbb{C} \tag{4.20}$$

and regard $\mathbb{Z}^3$ as the usual composition of two fcc lattices (cf. Figure 5) then we may associate a cross-ratio with a quadrilateral (P_1, P_2, P_3, P_4) formed by the vertices P_i of a tetrahedron contained in an elementary cube. If we denote by P_{i+4} the vertices of the second tetrahedon which are diagonally opposite P_i then another quadrilateral defined on the second tetrahedron is given by (P_5, P_6, P_7, P_8). The cross-ratio relation (2.6) is then expressed as

$$\mathsf{M}(\Phi_1, \Phi_2, \Phi_3, \Phi_4) = \mathsf{M}(\Phi_5, \Phi_6, \Phi_7, \Phi_8), \tag{4.21}$$

where again $\Phi_i = \Phi(P_i)$. This one equation implies any of the other equations obtained by a different choice of the (labelled) quadrilateral (P_1, P_2, P_3, P_4).

5 Cauchy problems. Connections with conjugate lattices and the discrete Darboux system

The multi-ratio conditions $\mathsf{M}_{\mathsf{O}} = -1$ and $\mathsf{M}_{\mathsf{CO}} = 1$ must be compatible since they represent integrable BKP lattices. However, their canonical representation (4.5), (4.14) constitutes an overdetermined set of equations for the slope functions α, β, γ. Hence, it may be suspected that the well-determined system (4.14) by itself is an integrable system and (4.5) represents an admissible constraint which is 'in involution'. Here, we show that this is indeed the case and discuss the implications of this observation.

5.1 The discrete Darboux system

In terms of the new dependent variables

$$\chi^{(1)} = \alpha_1, \quad \chi^{(2)} = \beta_2, \quad \chi^{(3)} = \gamma_3, \tag{5.1}$$

the multi-ratio conditions (4.14) are conveniently expressed as

$$\left(\frac{\chi_i^{(k)}-\chi^{(i)}}{\chi_i^{(k)}-\chi_k^{(i)}}\right)_l\left(\frac{\chi_i^{(k)}-\chi_k^{(i)}}{\chi_i^{(k)}-\chi^{(i)}}\right)=\left(\frac{\chi_i^{(l)}-\chi^{(i)}}{\chi_i^{(l)}-\chi_l^{(i)}}\right)_k\left(\frac{\chi_i^{(l)}-\chi_l^{(i)}}{\chi_i^{(l)}-\chi^{(i)}}\right), \tag{5.2}$$

where here and in the remainder of this paper $i,k,l\in\{1,2,3\}$ are assumed to be distinct. These may be satisfied identically by introducing potentials $H^{(i)}$ according to

$$\frac{H_k^{(i)}}{H^{(i)}}=\frac{\chi_i^{(k)}-\chi^{(i)}}{\chi_i^{(k)}-\chi_k^{(i)}}. \tag{5.3}$$

The latter are now interpreted as linear equations for $\chi^{(i)}$ and brought into the form

$$\Delta_k\chi^{(i)}=\rho^{(ik)}(\chi_i^{(k)}-\chi_k^{(i)}), \tag{5.4}$$

where Δ_k denote difference operators defined by $\Delta_k f=f_k-f$ and

$$\rho^{(ik)}=\frac{\Delta_k H^{(i)}}{H^{(i)}}. \tag{5.5}$$

Note that the systems (5.2) and (5.4) are completely equivalent. The linear system (5.4) is readily shown to be compatible if and only if the coefficients $H^{(i)}$ obey the nonlinear discrete equations

$$\Delta_{ik}H^{(l)}=\frac{\Delta_k H_l^{(i)}}{H_l^{(i)}}\Delta_i H^{(l)}+\frac{\Delta_i H_l^{(k)}}{H_l^{(k)}}\Delta_k H^{(l)} \tag{5.6}$$

with $\Delta_{ik}=\Delta_i\Delta_k$. These constitute a well-known integrable discretization of the Darboux equations defining conjugate coordinates in $\mathbb{R}^3$ [4, 13]. Accordingly, the multi-ratio conditions (4.14) represent a Möbius invariant avatar of the discrete Darboux system.

The discrete Darboux system governs conjugate lattices ('discrete conjugate coordinates') in $\mathbb{R}^3$ [3, 15]. Thus, consider a three-dimensional quadrilateral lattice in a three-dimensional Euclidean space, that is a map

$$\boldsymbol{r}:\mathbb{Z}^3\to\mathbb{R}^3. \tag{5.7}$$

Conjugate lattices are defined by the requirement that all elementary quadrilaterals be planar. In analytical terms, this means that the position vector $\boldsymbol{r}$ of a conjugate lattice obeys discrete 'hyperbolic' equations of the form

$$\Delta_{ik}\boldsymbol{r}=\rho^{(ik)}\Delta_i\boldsymbol{r}+\rho^{(ki)}\Delta_k\boldsymbol{r}. \tag{5.8}$$

Their compatibility conditions lead to the parametrization (5.5) of the coefficients $\rho^{(ik)}$ and to the discrete Darboux system (5.6) for the potentials $H^{(i)}$. It is then natural to introduce functions $\chi^{(i)}$ satisfying (5.4) which play the role of 'adjoint eigenfunctions' [23]. These may again be regarded as one function χ defined on the face centres $\mathbb{F}$ of the cubic lattice $\mathbb{Z}^3$. Thus, the preceding may be summarized in the following manner:

Theorem 6 (A Möbius invariant form of the discrete Darboux system). *The discrete Darboux system (5.6) governing conjugate lattices in* $\mathbb{R}^3$ *is equivalent to the multi-ratio conditions* $\mathsf{M}_{\mathsf{CO}} = 1$ *imposed on the adjoint eigenfunction* $\chi : \mathbb{F} \to \mathbb{R}$.

The geometric significance of the adjoint eigenfunctions is the following. If $\boldsymbol{r}$ represents a conjugate lattice then the relations

$$\Delta_i \tilde{\boldsymbol{r}} = \chi^{(i)} \Delta_i \boldsymbol{r} \tag{5.9}$$

are compatible if and only if the coefficients $\chi^{(i)}$ are solutions of the linear system (5.4). In this case, the lattice $\tilde{\boldsymbol{r}}$ constitutes another conjugate lattice known as a discrete Combescure transform [23] of the conjugate lattice $\boldsymbol{r}$. Discrete Combescure transforms are defined by the geometric property that their edges are parallel to those of the original conjugate lattice. In fact, any two lattices which are 'parallel', that is are related by (5.9), are necessarily conjugate lattices and the dilations $\chi^{(i)}$ obey the multi-ratio conditions (5.2). Accordingly, it has been established that the slope function σ associated with a BKP lattice may also be interpreted as a dilation function defining parallel conjugate lattices.

5.2 A well-posed Cauchy problem

Having shown that the multi-ratio conditions $\mathsf{M}_{\mathsf{CO}} = 1$ constitute an integrable system on their own, we now investigate in what sense the constraint $\mathsf{M}_{\mathsf{O}} = -1$ is admissible. A key observation is that the multi-ratios M_6 and M_8 are related by a simple identity. Consider a cubo-octahedron enclosed by six cubo-octahedra and eight octahedra. The multi-ratios which are associated with the six square faces of the central cubo-octahedron but are defined on the six neighbouring cubo-octahedra and the multi-ratios corresponding to the eight octahedra are algebraically dependent. In terms of the slope functions, this dependency is expressed as

$$\begin{gathered}
\mathsf{M}(\alpha,\beta_2,\alpha_2,\gamma_{23},\alpha_{23},\beta_{23},\alpha_3,\gamma_3)\,\mathsf{M}(\gamma_{13},\alpha_{11},\beta_{12},\alpha_{112},\gamma_{123},\alpha_{1123},\beta_{123},\alpha_{113})\\
\mathsf{M}(\beta,\gamma_3,\beta_3,\alpha_{13},\beta_{13},\gamma_{13},\beta_1,\alpha_1)\,\mathsf{M}(\alpha_{12},\beta_{22},\gamma_{23},\beta_{223},\alpha_{123},\beta_{1223},\gamma_{123},\beta_{122})\\
\mathsf{M}(\gamma,\alpha_1,\gamma_1,\beta_{12},\gamma_{12},\alpha_{12},\gamma_2,\beta_2)\,\mathsf{M}(\beta_{23},\gamma_{33},\alpha_{13},\gamma_{133},\beta_{123},\gamma_{1233},\alpha_{123},\gamma_{233})\\
= \qquad (5.10)\\
\mathsf{M}(\alpha,\beta_2,\gamma,\alpha_1,\beta,\gamma_3)\,\mathsf{M}(\alpha_{123},\beta_{1223},\gamma_{123},\alpha_{1123},\beta_{123},\gamma_{1233})\\
\mathsf{M}(\beta_1,\alpha_1,\gamma_1,\beta_{12},\alpha_{11},\gamma_{13})\,\mathsf{M}(\alpha_{23},\beta_{23},\gamma_{233},\alpha_{123},\beta_{223},\gamma_{23})\\
\mathsf{M}(\gamma_2,\beta_2,\alpha_2,\gamma_{23},\beta_{22},\alpha_{12})\,\mathsf{M}(\beta_{13},\gamma_{13},\alpha_{113},\beta_{123},\gamma_{133},\alpha_{13})\\
\mathsf{M}(\alpha_3,\gamma_3,\beta_3,\alpha_{13},\gamma_{33},\beta_{23})\,\mathsf{M}(\gamma_{12},\alpha_{12},\beta_{122},\gamma_{123},\alpha_{112},\beta_{12}).
\end{gathered}$$

Accordingly, if any seven multi-ratio conditions $\mathsf{M}_{\mathsf{O}} = -1$ associated with the octahedra are satisfied then the remaining eighth likewise holds provided that

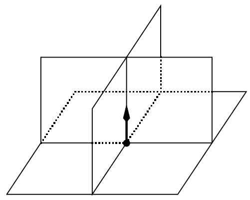

Figure 9: An eight-face complex

$\mathsf{M}_{\mathsf{CO}} = 1$ on the relevant cubo-octahedra. In other words, the multi-ratio condition $\mathsf{M}_{\mathsf{O}} = -1$ 'propagates' along the diagonals of the lattice. Specifically, if we choose, for instance, the representatives

$$\begin{aligned} \mathsf{M}(\alpha_1, \beta_2, \alpha_{12}, \gamma_{23}, \alpha_{123}, \beta_{23}, \alpha_{13}, \gamma_3) &= 1 \\ \mathsf{M}(\beta_2, \gamma_3, \beta_{23}, \alpha_{13}, \beta_{123}, \gamma_{13}, \beta_{12}, \alpha_1) &= 1 \\ \mathsf{M}(\gamma_3, \alpha_1, \gamma_{13}, \beta_{12}, \gamma_{123}, \alpha_{12}, \gamma_{23}, \beta_2) &= 1, \end{aligned} \tag{5.11}$$

which may be regarded as evolution equations in the $(1,1,1)$-direction, then the constraint

$$\mathsf{M}(\alpha, \beta_2, \gamma, \alpha_1, \beta, \gamma_3) = -1 \tag{5.12}$$

is preserved by the evolution (5.11) by virtue of the identity (5.10).

It is now convenient to identify the construction of the slope function σ with filling the cubic lattice $\mathbb{Z}^3$ with faces. Thus, if σ is known on (the centre of) a face then we insert this face into the lattice. Accordingly, if σ is known on five faces of an elementary cube then its value on the remaining face is determined by the multi-ratio condition $\mathsf{M}_{\mathsf{O}} = -1$ and we complete the cube by inserting the remaining face. Similarly, a multi-ratio condition $\mathsf{M}_{\mathsf{CO}} = 1$ may be visualized by an eight-face complex of the type displayed in Figure 9. Thus, if σ is known on any seven faces of this complex then the corresponding multi-ratio condition $\mathsf{M}_{\mathsf{CO}} = 1$ determines its value on the remaining face which is then inserted into the lattice. It is also natural to identify the cubo-octahedra with their centres which constitute the vertices of the cubic lattice. Hence, whenever a multi-ratio condition defined on a cubo-octahedron is known to be satisfied then we attach an arrow to the corresponding central vertex as indicated in Figure 9.

Natural Cauchy data associated with the lattice equations $\mathsf{M}_{\mathsf{CO}} = 1$ are given by

$$\begin{array}{ll} \alpha(n_1, n_2, 0), & \alpha(n_1, 0, n_3) \\ \beta(n_1, n_2, 0), & \beta(0, n_2, n_3) \\ \gamma(n_1, 0, n_3), & \gamma(0, n_2, n_3). \end{array} \tag{5.13}$$

A clipping of the corresponding 'initial state' of the lattice in the first octant is shown in Figure 10(a). The faces of the planes $n_i = 0$ and $n_i = 1$ may then be inserted iteratively on use of the multi-ratio conditions $\mathsf{M}_{\mathsf{CO}} = 1$, that is

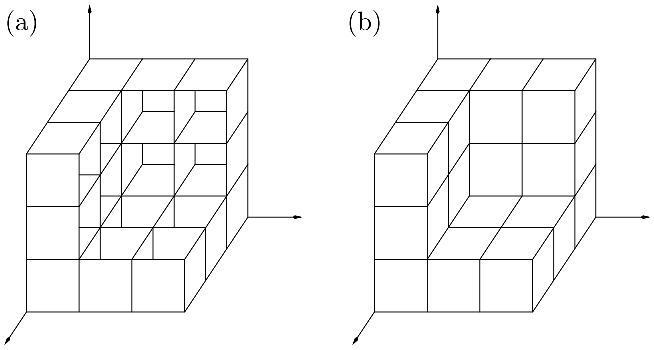

Figure 10: Cauchy data for $\mathsf{M}_{\mathsf{CO}} = 1$

eight-face complexes. The resulting intermediate state of the lattice is depicted in Figure 10(b). The additional data constructed in this manner read

$$\begin{aligned} &\alpha(0,n_2,n_3), \quad \alpha(1,n_2,n_3)\\ &\beta(n_1,0,n_3), \quad \beta(n_1,1,n_3)\\ &\gamma(n_1,n_2,0), \quad \gamma(n_1,n_2,1). \end{aligned} \tag{5.14}$$

It is now readily verified that the entire $\mathbb{Z}^3$ lattice may be filled with faces by means of eight-face complexes. Moreover, in the process each vertex becomes endowed with three arrows corresponding to three multi-ratio conditions per octahedron. Consequently, the multi-ratio conditions $\mathsf{M}_{\mathsf{CO}} = 1$ are satisfied throughout the lattice.

The preceding construction implies that we may choose Cauchy data of the form (5.13) *and* (5.14) as long as these allow for $\mathsf{M}_{\mathsf{CO}} = 1$ wherever applicable on the region Γ bounded by the planes $n_i = 0$ and $n_i = 1$. We now specialize these data in such a way that, in addition, the multi-ratio condition $\mathsf{M}_{\mathsf{O}} = -1$ holds on Γ. Thus, we arbitrarily prescribe the slope function on the boundaries of three square cylinders subject to the multi-ratio condition associated with the one central cube enclosed by these cylinders, that is (cf. Figure 11(a))

$$\begin{aligned} &\alpha(\epsilon,n_2,0), \quad \alpha(\epsilon,0,n_3)\\ &\beta(n_1,\epsilon,0), \quad \beta(0,\epsilon,n_3), \quad \epsilon = 0,1\\ &\gamma(n_1,0,\epsilon), \quad \gamma(0,n_2,\epsilon) \end{aligned} \tag{5.15}$$

and

$$\mathsf{M}(\alpha,\beta_2,\gamma,\alpha_1,\beta,\gamma_3)|_{(n_1,n_2,n_3)=(0,0,0)} = -1. \tag{5.16}$$

By virtue of the multi-ratio condition $\mathsf{M}_{\mathsf{O}} = -1$, the slope function is then uniquely determined on the faces inside the square cylinders. Moreover, if we choose additional data

$$\alpha(n_1 \neq \epsilon, n_2 \neq 0, 0), \quad \beta(0, n_2 \neq \epsilon, n_3 \neq 0), \quad \gamma(n_1 \neq 0, 0, n_3 \neq \epsilon) \tag{5.17}$$

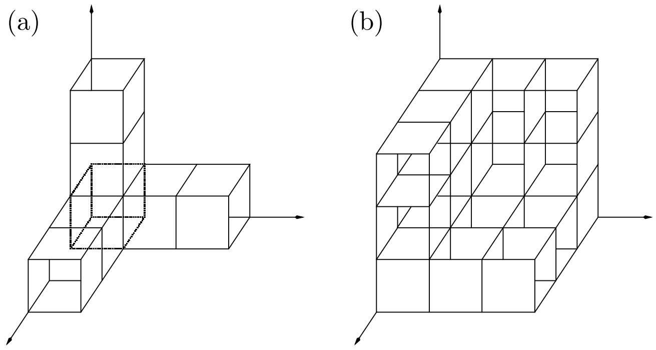

Figure 11: Cauchy data for $\mathsf{M}_{\mathsf{CO}} = 1$ and $\mathsf{M}_{\mathsf{O}} = -1$

as indicated in Figure 11(b), then iterative application of the multi-ratio conditions $\mathsf{M}_{\mathsf{CO}} = 1$ and $\mathsf{M}_{\mathsf{O}} = -1$ determines the slope function on Γ. By construction, all relevant multi-ratio conditions are satisfied on Γ. The data specified in this manner may now be used as Cauchy data of the form (5.13), (5.14) in order to construct a unique slope function σ satisfying $\mathsf{M}_{\mathsf{CO}} = 1$ throughout the lattice. Moreover, since the multi-ratio condition $\mathsf{M}_{\mathsf{O}} = -1$ propagates in all diagonal directions, it likewise holds everywhere. Accordingly, the following theorem obtains:

Theorem 7 (A well-posed Cauchy problem for BKP slope lattices). *Cauchy data of the form (5.15)-(5.17) uniquely determine a slope function σ obeying $\mathsf{M}_{\mathsf{CO}} = 1$ and $\mathsf{M}_{\mathsf{O}} = -1$.*

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